Homogeneous Systems and Independence

If we let $\vec{b} = \vec{0}$ in the matrix equation $A\vec{x} = \vec{b}$, the system is called homogeneous.

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= 0 \\
\vdots \quad\quad \vdots \quad\quad \vdots \quad\quad\quad\quad \vdots \quad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= 0
\end{aligned} \qquad \textbf{(4)}$$

$$A\vec{x} = \vec{0}$$

Remark

Unlike systems of the form $A\vec{x} = \vec{b}$ which can be inconsistent, the homogeneous system $A\vec{x} = \vec{0}$ is always consistent since $\vec{x} = \vec{0}$ is a trivial solution to the system. The space of solutions of $A\vec{x} = \vec{0}$ is called the **Nullspace** of A, or simply the zeros of the matrix A.

Example 1

Find all zeros of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & 0 \\ 1 & 6 & 11 \end{bmatrix}$ (solutions of $A\vec{x} = \vec{0}$).

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ -2 & -3 & 0 & 0 \\ 1 & 6 & 11 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 4 & 8 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -9 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & -16 & 0 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -9 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore, the only zero of the matrix A is the origin $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, and the nullspace of A is of dimension zero.

Example 2

Find all zeros of the matrix $B = \begin{bmatrix} 1 & 2 & 5 \\ -1 & 0 & -1 \\ 1 & 6 & 13 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 & 5 \\ -1 & 0 & -1 \\ 1 & 6 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 5 \\ 0 & 2 & 4 \\ 0 & 4 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 5 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 5 \\ -1 & 0 & -1 \\ 1 & 6 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 5 \\ 0 & 2 & 4 \\ 0 & 4 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 4 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, the zeros of B are

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ -2z \\ z \end{pmatrix} = z \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix},$$

which is a line through the origin in $\mathbb{R}^3$. In this example the nullspace of B is of dimension 1.

Definition

A set of vectors $\{v_1, v_2, v_3, \ldots, v_n\}$ is called **linearly independent** if the homogeneous system

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$$

has only the trivial solution, namely $\vec{c} = \vec{0}$. In other words, independent vectors don't collapse to zero if we take a linear combination of them unless all the c's are equal to zero. Otherwise, the vectors are **linearly dependent**.

Example 1 Are the vectors $v_1 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$ independent?

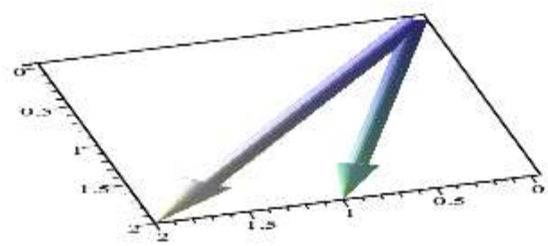
$$c_1 \cdot \begin{pmatrix} 1 \\ 5 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 2 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 5 & 8 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

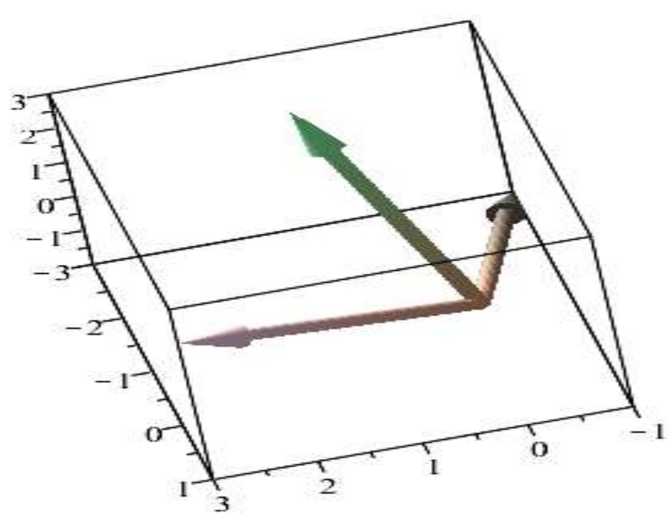
Solution: $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and therefore vectors v_1 and v_2 are independent.

Remark:

In $\mathbb{R}^2$, two vectors are independent as long as they are not on the same line.



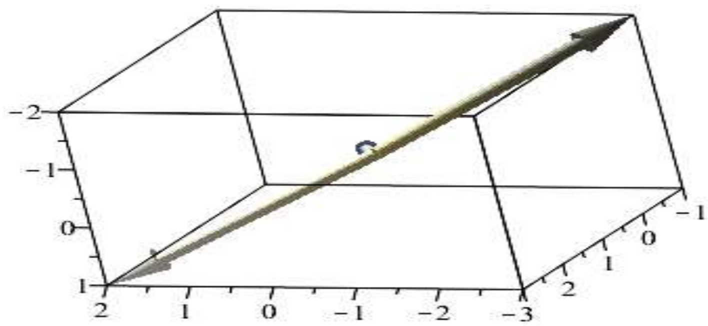
In $\mathbb{R}^3$, three vectors are independent as long as no two are on the same line, and not all three are on the same plane.



Example 2

The three vectors below $u = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $v = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}$, and $w = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$ seem to be **dependent**.

Investigate whether the statement is true or not.



$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 2 & -3 & -1 & 0 \\ 3 & -1 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 5 & 5 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solutions: $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -c_3 \\ -c_3 \\ c_3 \end{bmatrix}$

Test

$$-c_3 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - c_3 \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -c_3 \\ -2c_3 \\ -3c_3 \end{pmatrix} + \begin{pmatrix} 2c_3 \\ 3c_3 \\ c_3 \end{pmatrix} + \begin{pmatrix} -c_3 \\ -c_3 \\ 2c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Homework

1. Are the vectors $u = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$, $v = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$, and $w = \begin{pmatrix} 7 \\ 10 \\ 12 \end{pmatrix}$ independent? Explain.

2. Are the vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 3 \\ 5 \\ 0 \end{pmatrix}$ independent? Justify your answer.

3. What is the maximum number of independent vectors we can have in $\mathbb{R}^n$?

4. Show that if the vectors $v_1 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 4 \\ 6 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 10 \\ 15 \\ -1 \end{pmatrix}$ are **dependent**, one of them must be a linear combination of the other two.

$$\begin{bmatrix} 1 & 0 & -1 \\ 3 & 6 & 15 \\ 2 & 4 & 10 \end{bmatrix} \xrightarrow{RREF} \cdots \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad \cdots$$

5. Find the zeros, solutions to $A\vec{x} = \vec{0}$, of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 3 & 6 & 9 \end{bmatrix}.$$

6. Find the zeros of the matrix

$$A = \begin{bmatrix} 1 & -2 & -1 & 3 & 5 \\ -1 & 3 & 4 & 0 & 6 \\ 7 & -1 & -16 & 12 & 2 \end{bmatrix}.$$